ALGEBRAIC ON MAGIC SQUARE OF ODD ORDER n

Mahyuddin

Department of Mathematics, University of North Sumatera, Medan 20155, Indonesia
mahyunst@yahoo.com

Abstract

In this paper we described the relation between a magic square of odd order n and a group, and their properties. By the modulo number n, we construct entries for each table from initial table of magic square with large number n^2. Generalization of the underlying ideas are presented, we have unique group, and we also prove variants of the main results for magic cubes.

Keywords: entry, array, algorithm, magic cubes, group.

1. Introduction

According to the book of W. S. Andrews [2], the study of magic squares is quite old and dates back to ancient Tibet, to 12th century China, to 9th century Arab astrologers and perhaps much further. Speculation of them might even be prehistoric.

In this paper, we shall see old procedure can product unique magic square based on a group of a set in modulo n. Objectives are find new magic square which it has different procedure compares with old. Therefore, we get a new procedure to product new magic square. Of course, this paper organized by first we defined the magic square with all conditions to product it, and based on conditions we result a procedure on odd order n. Next section, we modify all entries of magic square on modulo n and we test all conditions of magic square. Finding new procedure based on these magic square with all entries on modulo n, and we forse a simple group on a set Z_n with a binary operation, and based on we find new table of magic square. Are all magic square of odd order n satisfy all conditions of magic square?

2. The Magic Squares

By a magic square of order n, also called an $n \times n$ magic square, we mean an $n \times n$ square array $A = (a_{ij})$, $0 \le i, j \le n-1$, of positive integers such that

i. each of the integers from 1 to n^2 inclusive occurs exactly once among the entries of A,

ii. for $0 \leq i \leq n-1$, the sum $\sum_{j=0}^{n-1} a_{ij}$ is independent of *i*,

iii. for $0 \leq j \leq n-1$, the sum $\sum_{j=0}^{n-1} a_{ij}$ is independent of *j*,

iv. the sums $\sum_{i=0}^{n-1} a_{ii}$ dan $\sum_{i=0}^{n-1} a_{i,n-i-1}$ are equal to the sums given in (ii), and therefore to those in (iii) as well.

Table 1: 3×3 magic square

8	1	6
3	5	7
4	9	2

Table 2: 4×4 magic square

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

For example, let *A* be the 3×3 magic square in Table 1 and let *B* be the 4×4 magic square in Table 2. The product square *A*B* has order 12, the product of the orders of *A* dan *B*. Let *G* be an abelian group and choose an element *u* of *G* once and for all. Denote by *A*(*G*) the set of all square arrays of lements of *G*, the size of the arrays being arbitrary. If *A* = (a_{ij}), $0 \leq i, j \leq m-1$, and *B* = (b_{kl}), $0 \leq k, l \leq n-1$ are elements of *A*(*G*) of sizes $m \times m$ and $n \times n$ respectively, then *A*B* will be the $mn \times mn$ matrix *E* = ($e_{\alpha\beta}$), (α, β)-the entry.

A set *S* closed under an assosiative operation is called a *semigroup.*, and we call *monoid* if operation with an identity element. Therefore we have the some result [1]:

Lemma 1. *Let G be an abelian group. The set A(G) of all square arrays with entires in G is a monoid with identity element u with respect to the operation * defined by*

$$e_{\alpha\beta} = m^2(b_{kl} + u) + a_{ij} \tag{1}$$

and

$$(\alpha, \beta) = m(k, l) + (i, j) \tag{2}$$

Let *S* be a monoid with identitiy element *S* with respect to an operation $\circ$. We say that *S* is *left cancellative* if for any elements *a*, *b*, *c* of *S*, the identity $a \circ b = a \circ c$ implies *b* – *c*. Similarly, we say that *S* is *right cancellative* for the other side. In this case, the entries of a magic square of order *n* run from 0 to n^2-1 instead of from 1 to n^2, and we take *u* = 0 instead of *u* = -1.

Let A is $m \times m$, B is $n \times n$ and C is $p \times p$. If $A * C - B * C$ then we must have $mp - np$ and therefore $m = n$. Therefore, we have $m^2(c_{kl} - 1) + a_{ij} = m^2(c_{kl} - 1) + b_{ij}$ for $0 \le i, j \le m-1$ and $0 \le k, l \le n-1$, which implies that $a_{ij} = b_{ij}$ for all i, j. Therefore $A*B - B*C$ implies A = B. If $A*B - A*C$ then we must have $mn-mp$ and therefore $n-p$. It follows that $m^2(b_{kl} + 1) + a_{ij} = m^2(c_{kl} - 1) + a_{ij}$ for $0 \le i, j \le m-1$ and $0 \le k, l \le n-1$, which implies that $m^2(b_{kl} - c_{kl}) = 0$ for all k,l. It follows that $b_{kl} = c_{kl}$ for all k, l, and equation $A*B - A*C$ implies $B - C$.

Lemma 2. *Let G be an abelian group and let u be an element of G. Then the monoid (A(G),*,u) is right cancellative.*

Lemma 3. *Let G be an abelian group and let u be an element of G. Then the monoid (A(G),*,u) is left cancellative if and only if the group G is torsion-free.*

3. The Odd Order

Let we consider the magic square of odd order n, i.e. $n = 3, 5, 7, 9, \ldots$. All conditions (i), (ii), (iii) and (iv) above give entries of a magic square of odd order n based on following steps:

Step 1: Set $i = (n-1)/2$, $j = 0$, and $k = 1$.

Step 2: Do While $k \le n^2$

Step2a: If $j = -1$ Then

If $i - n$ Then $i - i - 1$ and $j - j + 2$ Else $j - n - 1$

Step2b: If $i - n$ Then $i = 0$

Step2c: If $a_{ij} \ge 0$ Then $j - j + 2$ and $i - i - 1$

Step 2d: $a_{ij} - k$

Step 2e: $j - j - 1$, $i - i + 1$ and $k - k + 1$

Step 3: End Do

For example,

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

,

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

and

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

are magic squares of odd order n, $n = 5, 7, 9$.

If we modify condition (i) of magic square so that the entries of a magic square of odd order n run from 1 to n^2 modulo n instead of from 1 to n^2, we get a difference between a column (left side) and next column (to right side) is 2 in modulo n and a difference between a row and next row from top to down is 1 in modulo n. Therefore, we define function $f: Z_n \times Z_n \to Z_n$ or a binary operation on Z_n defined by $f(r,c) = 2 + r + 2c$, for all r, c in Z_n, r and c represent row and column of table, respectively. For example, $n = 3, 5, 7$ and 9 we have

2	1	0
0	2	1
1	0	2

,

2	4	1	3	0
3	0	2	4	1
4	1	3	0	2
0	2	4	1	3
1	3	0	2	4

,

2	4	6	1	3	5	0
3	5	0	2	4	6	1
4	6	1	3	5	0	2
5	0	2	4	6	1	3
6	1	3	5	0	2	4
0	2	4	6	1	3	5
1	3	5	0	2	4	6

and

2	4	6	8	1	3	5	7	0
3	5	7	0	2	4	6	8	1
4	6	8	1	3	5	7	0	2
5	7	0	2	4	6	8	1	3
6	8	1	3	5	7	0	2	4
7	0	2	4	6	8	1	3	5
8	1	3	5	7	0	2	4	6
0	2	4	6	8	1	3	5	7
1	3	5	7	0	2	4	6	8

By each table of Z_n with f, $n = 3k$, $k = 1, 3, 5, 7, 9, \ldots$, we have result of condition (iv) of a magic square, i. e. $\sum_{i=0}^{n-1} a_{ii}$ (it called sltdd or sums of left-top-down diagonal) is not equal to

$\sum_{i=0}^{n-1} a_{i,n-i-1}$ (slbud or sums of left-buttom-up diagonal) or other condition (sr or sums of rows and sc or sums of column), see Appendix.

A set Z_n with a binary operation f is not an assosiative operation, and then (Z_n,f) is not a group.

Let we have a procedure to product new magic square (it called New MS), may be, so that we get a set Z_n with a binary operation satisfy all conditions of group, i.e.,

Step 1: Copy $a_{n-1,j}$ to first column of table

Step 2: Copy $a_{(n-1)/2,j}$ to second column of table

Step 3: Copy $a_{0,j}$ to third column of table

Step 4: Set $i = 3, x = 0, y = 0$

Step 5: Do While $i < n$

Step 5a: If (i modulo $2 = 0$) Then

$x - x+1$

Copy a_{xj} to a_{ij}

Else

$y = y + 1$

Copy $a_{(n-1)/2+y,j}$ to a_{ij}

Step 6: End Do

For example, for the entries of pair table, we run 1 to n^2 and 1 to n^2 modulo n, $n = 3, 5, 7, 9$, respectively,

6	1	8
7	5	3
2	9	4

0	1	2
1	2	0
2	0	1

15	1	17	8	24
16	7	23	14	5
22	13	4	20	6
3	19	10	21	12
9	25	11	2	18

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

28	1	30	10	39	19	48
29	9	38	18	47	27	7
37	17	46	26	6	35	8
45	25	5	34	14	36	16
4	33	13	42	15	44	24
12	41	21	43	23	3	32
20	49	22	2	31	11	40

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5

and

45	1	47	12	58	23	69	34	80
46	11	57	22	68	33	79	44	9
56	21	67	32	78	43	8	54	10
66	31	77	42	7	53	18	55	20
76	41	6	52	17	63	19	65	30
5	51	16	62	27	64	29	75	40
15	61	26	72	28	74	39	4	50
25	71	36	73	38	3	49	14	60
35	81	37	2	48	13	59	24	70

0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3
5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7

By table of New MS, we have a table represent a binary operation, where it defined by function $g : Z_n \times Z_n \to Z_n$ or $g(a,b) = a+b$, for all a, b in Z_n, and then we have exactly a set Z_n with g construct a simple group (G mod n) where $\sum_{i=0}^{n-1} a_{i,n-i-1}$ is $2\sum_{i=0}^{n-1} a_{ii}$ or $2\sum_{j=0}^{n-1} a_{ij}$ or $2\sum_{j=0}^{n-1} a_{ij}$.

Based on first table, for n = 5, 7, 11, ..., New MS satisfies conditons (i), (ii), (iii), and (iv) of magic square, it excepts a part of condition (iv), i. e $\sum_{i=0}^{n-1} a_{i,n-i-1}$, whereas for n = 3 and 9 we get two special cases, where all conditions of magic square hold. So, there exist two simple group of Z_n, n = 3, and 9, with g and both have pure magic square.

We let M_2 denote the set of all magic squares, include New MS of odd order n = 3 and 9. Let we form the magic square $A*B$ with A is New MS n = 3 and B is n = 3 or 4 from old magic square. as below.

51	46	53	6	1	8	69	64	71
52	50	48	7	5	3	70	68	66
47	54	49	2	9	4	65	72	67
60	55	62	42	37	44	24	19	26
61	59	57	43	41	39	25	23	21
56	63	58	38	45	40	20	27	22
15	10	17	78	73	80	33	28	35
16	14	12	79	77	75	34	32	30
11	18	13	74	81	76	29	36	31

Now locate the square in B (Table 1 or 2) which contains the number 1 and place a copy of A in the correspondeing square of the frame we have just constructed. Now locate the square in B containing 2 and in the corresponding square, count out the next 9 numbers in the same pattern. It is the same to say that one adds 9 to all of the entries of A and places the result in the box corresponding to the position of the 2 in B. Next one finds the 3 of B and counts out the next 9 numbers in the corresponding place in the frame. Continuing in this way, we eventually get the magic square completely by 4 methods.

69	64	71	6	1	8	51	46	53
70	68	66	7	5	3	52	50	48
65	72	67	2	9	4	47	54	49
24	19	26	42	37	44	60	55	62
25	23	21	43	41	39	61	59	57
20	27	22	38	45	40	56	63	58
33	28	35	78	73	80	15	10	17
34	32	30	79	77	75	16	14	12
29	36	31	74	81	76	11	18	13

53	46	51	8	1	6	71	64	69
48	50	52	3	5	7	66	68	70
49	54	47	4	9	2	67	72	65
62	55	60	44	37	42	26	19	24
57	59	61	39	41	43	21	23	25
58	63	56	40	45	38	22	27	20
17	10	15	80	73	78	35	28	33
12	14	16	75	77	79	30	32	34
13	18	11	76	81	74	31	36	29

and

6	1	8	132	127	134	123	118	125	33	28	35
7	5	3	133	131	129	124	122	120	34	32	30
2	9	4	128	135	130	119	126	121	29	36	31
105	100	107	51	46	53	60	55	62	78	73	80
106	104	102	52	50	48	61	59	57	79	77	75
101	108	103	47	54	49	56	63	58	74	81	76
69	64	71	87	82	89	96	91	98	42	37	44
70	68	66	88	86	84	97	95	93	43	41	39
65	72	67	83	90	85	92	99	94	38	45	40
114	109	116	24	19	26	15	10	17	141	136	143
115	113	111	25	23	21	16	14	12	142	140	138
110	117	112	20	27	22	11	18	13	137	144	139

The product square $A*B$ has order 12, the product of the orders of A and B. It is convenient to have an analytic expression for this operation as (1) and (2) from Lemma 1. For $n = 3$ and 9 we see easy that $m - n$, a binary operation (1) to be

$$e_{\alpha\beta} = n^2(b_{kl} + u) + a_{ij}$$

and therefore Lemma 1, 2 and 3 hold for $n = 3$ and 9 of new magic square.

3. Conclusion

By the simple group Z_n we can product a procedure to construc a new magic square which they satisfies all conditions of magic square for $n = 3$ and 9, and these new magic squares so satisfy Lemma 1, 2 and 3.

References

1. Allan Adler and S. Y. Robert Li, "Magic *N*-Cubes and prouhet Sequences," *American Mathematical Monthly* **84** (1997), 618-627.
2. W. S. Andres, *Magic Squares and Cubes*, Dover, New York, 1960.

Appendix

Table 3: Result of computation of all conditon magic square

N	N^2	Case	Sr	Sc	Sltdd	Slbud
3	9	MS	15	15	15	15
		Mod n	3	3	6	3
		G mod n	3	3	3	6
		New MS	15	15	15	15
5	25	MS	65	65	65	65
		Mod n	10	10	10	10
		G mod n	10	10	10	20
		New MS	65	65	65	70
7	49	MS	175	175	175	175
		Mod n	21	21	21	21
		G mod n	21	21	21	42
		New MS	175	175	175	189
9	81	MS	369	369	369	369
		Mod n	36	36	45	36
		G mod n	36	36	36	72
		New MS	369	369	369	369
11	121	MS	671	671	671	671
		Mod n	55	55	55	55
		G mod n	55	55	55	110
		New MS	671	671	671	715
...						
99	9801	MS	485199	485199	485199	485199
		Mod n	4851	4851	4950	4851
		G mod n	4851	4851	4851	9702
		New MS	485199	485199	485199	489951
...						